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FLOPS AND EQUIVALENCES OF DERIVED CATEGORIES FOR THREEFOLDS WITH ONLY TERMINAL GORENSTEIN SINGULARITIES

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Abstract

The main purpose of this paper is to show that Bridgeland's moduli space of perverse point sheaves for certain flopping contractions gives the flops, and the Fourier-Mukai transform given by the birational correspondence of the flop is an equivalence between bounded derived categories.

1. Introduction

1.1 The minimal model program

One of the most important problems in birational geometry is the minimal model program (MMP). The main goal of the MMP is to find in each birational class of varieties some distinguished representatives (minimal models) which are "easier" to understand, then to use these minimal models to study the birational properties of varieties.

In dimension 2, satisfactory answers have been known for a long time. The procedure for producing a minimal model for X is repeatedly contracting a (-1)-curve. The final result of the MMP for a nonruled surface is a smooth surface such that it is minimal in the category of smooth surfaces (minimal in the classical sense), and its canonical bundle is *nef* (minimal in the sense of the MMP). In higher dimensions, the situation is much more complicated. Certain kinds of singularities are needed even if we start with a smooth variety. Besides singularities, we also need to consider flops and flips, which do not occur in dimension 2.

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Based on contributions from Reid, Mori, Kawamata, Kollár, Shokurov and others, the MMP program was completed in dimension 3 by Mori in 1988.

The proof of the MMP in dimension 3 uses a very careful analysis on two-dimensional Du Val singularities and threefold singularities. It is very difficult to generalize the proof along these lines to higher dimensions. A more conceptual proof is very desirable.

Flops can be considered as a sort of "birational surgery", an analogue of surgery in algebraic topology. A very natural and interesting question is what kind of invariants remain the same under flops. An example in this direction is that two birational nonsingular Calabi-Yau manifolds have the same Hodge numbers (see [3] for a result on Betti numbers, or [24] for a more general theorem). In dimension 3, this theorem was first proven in [14].

1.2 Flops and derived categories

Following Bondal-Orlov [5] and Bridgeland [8], it is plausible that the MMP may be understood in the context of derived categories. Given a variety X, the minimal model(s) might be viewed as some minimal triangulated subcategories inside $D^b(X)$. In this picture, it is very natural to view flops as taking different triangulated subcategories which are equivalent to one another, and flips as taking suitable fully faithful triangulated subcategories. There is considerable evidence to support this picture. A very important and interesting theorem to support this picture is a theorem by Bridgeland.

In [8] Bridgeland gives a moduli construction of smooth threefold flops. The moduli space he constructs is actually a fine moduli space. Furthermore, he is able to prove a result on the equivalence of derived categories by using techniques in [7] and [9]. As a corollary of his theorem, he proves again that two birational nonsingular Calabi-Yau threefolds have the same Hodge numbers. An interesting question is: Is Bridgeland's theorem true for singular varieties? In this paper we generalize his theorem to threefolds with terminal Gorenstein singularities. We remark here that these singularities are isolated hypersurface singularities (see [16] p. 169). The main theorem in our paper is:

Theorem 1.1. *Let X be a quasi-projective threefold with only terminal Gorenstein singularities and let $f : X \to Y$ be a flopping contraction. Denote by $W = W(X/Y)$ the distinguished component of the moduli space of perverse point sheaves $M(X/Y)$* (see Appendix A for the definitions). *Denote the canonical morphism* $W \to Y$ *by* $g : W \to Y$*. Then:*

(1) W *has only terminal Gorenstein singularities.*

(2) *The Fourier-Mukai type transform* $\Psi : D^b(W) \to D^b(X)$ *induced by the universal perverse point sheaves is an equivalence.*

(3) $W \to Y$ *is a flop of* $f : X \to Y$*.*

1.3 Reduction to the local cases

We outline the proof of this theorem in the subsequent subsections. First, a few comments on Fourier-Mukai type transforms. A Fourier-Mukai type transform F may not send $D^b(W)$ to $D^b(X)$ since X and W may be singular. However, the kernel we consider is $[\, \mathcal{I} \to \mathcal{O}_{W\times X}\,]$, where $\mathcal{I}$ is the universal perverse ideal sheaf and hence is flat over W. We show in Section 2 that such a kernel does define a transform $\Psi : D^b(W) \to D^b(X)$. Let $\{Y_i\}$ be an affine cover of Y. We pull back this universal perverse point sheaf to each Y_i. These kernels give Fourier-Mukai type transforms $\Psi_i : D^b(W_i) \to D^b(X_i)$.

We note that Theorem 1.1 is local in Y: Since the moduli space W is local in Y (see Remark A.8 in Appendix A), this is clear for part (1) and part (3) of Theorem 1.1. It is not obvious that part (2) is also local in Y since we can not check whether a functor is an equivalence or not locally. The next proposition shows that part (2) of the theorem is also local in Y. The main point of the proof is that Ψ has a right adjoint Φ.

Proposition 1.2 (see Proposition 3.2)**.** *Let the notation be as above. If there is an affine cover* $\{Y_i\}$ *of* Y *such that* $\Psi_i : D^b(W_i) \to D^b(X_i)$ *are equivalences of derived categories, then* $\Psi : D^b(W) \to D^b(X)$ *is also an equivalence of derived categories.*

1.4 Results from [8] and [9]

The argument in [8] uses the nonsingularity assumption in a significant way. The techniques used in his proof do not seem to generalize directly to singular varieties. Our idea is that instead of studying the singular threefold directly, we study a nonsingular fourfold, which is a smoothing, and see how much information about the singular threefold we can get from this smooth fourfold.

The starting point of our approach in this paper is the following theorem, which is a restatement of a combination of results in [8] and [9]. We sketch a proof in Appendix B for the reader's convenience.

Theorem 1.3 ([8] & [9]). *Let $f : X \to Y$ be a flopping contraction where X is an n-dimensional smooth quasi-projective variety and the dimension of every fiber is at most 1. Let W be the distinguished component of the moduli space of perverse point sheaves $M(X/Y)$. Assume $\dim(W \times_Y W) \leq n+1$. Then:*

(1) *W is smooth.*

(2) *The transform $D^b(W) \to D^b(X)$ induced by the universal perverse point sheaf is an equivalence.*

(3) *$W \to Y$ is a flop of $f : X \to Y$.*

The next corollary follows immediately from Theorem 1.3.

Corollary 1.4. *Let $f : X \to Y$ be a flopping contraction with X smooth. Then:*

(1) *If $\dim(X) = 3$, then the conclusions in Theorem 1.3 always hold.*

(2) *If $\dim(X) = 4$, every fiber of $f : X \to Y$ is of dimension at most 1, and $g : W \to Y$ does not contract any divisor to a point, then the conclusions in Theorem 1.3 hold.*

The next proposition is a combination of results in [9] and [8] as indicated by Bridgeland in the introduction in [8]. We shall not need this result in this paper.

Proposition 1.5 (see [8] & [9]). *With notation as in Theorem 1.3, assume that $\dim(X) = 3$. Then $M(X/Y) = W(X/Y)$.*

1.5 Relations between $W(X/Y)$ and $W(S/T)$ for a Cartier divisor $T \subset Y$

Let X be a variety with at worst terminal Gorenstein singularities. Assume that $\dim(X)$ is either 3 or 4 and $f : X \to Y$ satisfies the following two conditions:

(B.1) $\mathbf{R}f_*\mathcal{O}_X = \mathcal{O}_Y$.

(B.2) Every fiber of f is of dim ≤ 1.

Let $T \subset Y$ be an effective Cartier divisor; for simplicity we assume that it is an integral subscheme of Y. Let S be the preimage of T in X. Denote by $W_T = W(X/Y)_T$ the restriction of the moduli space $W(X/Y)$ to $T \subset Y$. The underlying philosophy of our approach is that:

(1) We find a smoothing $F : \mathcal{X} \to \mathcal{Y}$ of $f : X \to Y$.

(2) We relate the fiber of the moduli spaces $W(\mathcal{X}/\mathcal{Y})$ to the moduli space $W(X/Y)$.

The next proposition shows that (2) is possible.

Proposition 1.6 (see Proposition 4.4)**.** *With the notation as above, there is a canonical morphism $W(X/Y)_T \hookrightarrow M(S/T)$, which is an inclusion of components.*

Remark 1.7. It is also true that $M(X/Y)_T = M(S/T)$. We shall not need this stronger result in our paper.

1.6 Smoothing and smooth hyperplane sections

The following proposition shows that smoothing is always possible after passing to an affine cover.

Proposition 1.8 (see Proposition 5.2 for the precise statement)**.** *Let $X \to Y$ be a flopping contraction between threefolds where X has at worst terminal Gorenstein singularities. Then there is an affine cover $\{Y_i\}$ of Y such that each $f_i : X_i \to Y_i$ is a smoothable morphism.*

In the remainder of this subsection and the next subsection, we work over Y_i. We shall suppress the indices when no confusion is possible.

Let $F : \mathcal{X} \to \mathcal{Y}$ be a one-parameter deformation of $f : X \to Y$ such that $\mathcal{X}$ is nonsingular. Let $Y_{\text{sing}} = \{p_i : i = 1, \ldots, m\}$ be the finite set of singular points of Y. We also consider them as points of $\mathcal{Y}$. Let $\mathcal{T}$ be a general hyperplane section passing through $Y_{\text{sing}} \subset Y \subset \mathcal{Y}$. Denote by $\mathcal{S}$ the preimage of $\mathcal{T}$.

The following proposition enables us to use results on the smooth threefolds in [8].

Proposition 1.9 (see Proposition 5.3 for the precise statement)**.** *The hyperplane section $\mathcal{S}$ is nonsingular.*

1.7 Proofs of "no divisor is contracted to a point" and Theorem 1.1

Denote by $\mathcal{W}$ the distinguished component of the moduli space of perverse point sheaves for $F : \mathcal{X} \to \mathcal{Y}$. Denote by $G : \mathcal{W} \to \mathcal{Y}$ the natural birational morphism. We explain briefly how to prove that $G : \mathcal{W} \to \mathcal{Y}$ contracts no divisor to a point. Assume that there is a divisor contracted to a point, say p, by G. This point p must be one of the singular points in Y. Take a general hyperplane section $\mathcal{T}$ of $\mathcal{Y}$ passing through p. The preimage of $\mathcal{T}$, denoted by $\mathcal{S} \subset \mathcal{X}$, is smooth. By [8] the connected component of $W(\mathcal{S}/\mathcal{T}) \subset M(\mathcal{S}/\mathcal{T})$ is smooth. Every component W_j of $W(\mathcal{X}/\mathcal{Y})_{\mathcal{T}}$ is a component of $M(\mathcal{S}/\mathcal{T})$ by Proposition 1.6. The fiber $W(\mathcal{X}/\mathcal{Y})_{\mathcal{T}}$ is connected. Therefore, the distinguished component $W(\mathcal{S}/\mathcal{T})$ is the only component by the smoothness result. Since $W(\mathcal{S}/\mathcal{T}) \to \mathcal{T}$ is birational, it follows that the preimage of p is at most two-dimensional, a contradiction.

By Corollary 1.4, it follows that the Fourier-Mukai type transform $D^b(\mathcal{W}) \to D^b(\mathcal{X})$ is an equivalence and $\mathcal{W} \cong \mathcal{X}^+$.

By standard results on flops, it follows easily that $W \cong X^+$ is the flop and hence has only terminal Gorenstien singularities (see Section 6). This concludes the proof of part (1) and (3) in Theorem 1.1.

To prove $D^b(W) \cong D^b(X)$, more work is needed. Let $\Psi : D^b(\mathcal{W}) \to D^b(\mathcal{X})$ be the Fourier-Mukai type transform defined by the universal perverse point sheaf, i.e., the structure sheaf of the fiber product $\mathcal{W} \times_{\mathcal{Y}} \mathcal{X}$. Let i_0 be the inclusion morphism $W \to \mathcal{W}$ (see Proposition 4.4). Denote by $\Psi_0 : D^b(W) \to D^b(X)$ the Fourier-Mukai type transform defined by $\mathbf{L}i_0^* \mathcal{O}_{\mathcal{W} \times_{\mathcal{Y}} \mathcal{X}}$. Note that this Fourier-Mukai type transform is equivalent to the Fourier-Mukai type transform defined by the kernel $\mathcal{O}_{W \times_Y X}$ (see Proposition 4.4 and Corollary 4.5). We shall denote both of these two functors by Ψ_0. Denote by $\Phi : D^b(\mathcal{X}) \to D^b(\mathcal{W})$ the right adjoint to Ψ. This functor is also a Fourier-Mukai type transform (see Lemma 4.5 in [7]).

To complete the proof of part (2) in Theorem 1.1, we use the next proposition. The proof of this proposition is given in Section 6. The main point is to show that $\Psi(i_{0\,*}(-)) \cong i_{0\,*}(\Psi_0(-))$.

Proposition 1.10 (see Proposition 6.2)**.**

$$\Psi : D^b(\mathcal{W}) \cong D^b(\mathcal{X}) \implies \Psi_0 : D^b(W) \cong D^b(X).$$

Remark 1.11. Using the results in [19] and [20], our results imply the K-theories of coherent sheaves (i.e., G-theories) of X^+ and X are isomorphic.

1.8 Comments and further developments

Finally, we would like to say a few words on the limitation of the smoothing approach and our speculation on the possible generalizations.

It is well-known that quotient singularities in dimension ≥ 3 are rigid. Therefore our smoothing approach would not work for the most general threefold flops. To settle general three-dimensional flops using Bridgeland's approach, it seems that new ideas and techniques are needed. We speculate that algebraic stacks should play certain roles in the complete picture. Recently Kawamata proved an interesting result on n-dimensional toric flips and derived categories (see [12]). His result provides some evidence to support our speculation.

In the flips cases, D. Abramovich and I are working on some simple toric flips ([1]). In that case, we use the natural stack structure on threefolds in question instead of using deformations. We also plan to use the similar stack structure to extend our results to Q-Gorenstein case.

1.9 Plan of the paper

The plan of this paper is as follows. In Section 2 we present a few basic facts about the Fourier-Mukai type transforms. In Section 3, we explain how to reduce the proof to an affine Zariski neighborhood of Y. In Section 4, we prove several facts on the moduli space of perverse point sheaves. In Section 5, we give the proofs of lemmas on the deformation and general hyperplane sections needed for our proof. We give a proof on how to deduce the equivalence of derived categories in dimension 3 from the corresponding result in dimension 4 in Section 6.

The first appendix contains basic facts about triangulated categories and perverse coherent sheaves. All the material is taken from [8]. We sketch the proof of Theorem 1.3 in the second appendix. The proof is the same as the proof in [9].

1.10 Notation

All schemes T are schemes of finite type over $\mathbf{C}$. Denote by T^n the normalization of T. Denote by $D_{qc}(T)$ the derived category of the abelian category $Q\mathrm{coh}(T)$ of quasi-coherent $\mathcal{O}_T$-modules. Denote by $D^+(T)$ the full subcategory of $D_{qc}(T)$ consisting of complexes whose cohomology sheaves are bounded below and coherent. Denote by $D^-(T)$ the full subcategory of $D_{qc}(T)$ consisting of complexes whose cohomology sheaves are bounded above and coherent. Denote by $D^b(T)$ the full subcategory of $D_{qc}(T)$ consisting of complexes with bounded and coherent cohomology sheaves. Denote by $D^b_c(T)$ the full subcategory of $D^b(T)$ consisting of complexes whose cohomology sheaves are of proper support.

Let $f : T \to S$ be a projective birational morphism such that the conditions (B.1) and (B.2) are satisfied. We denote by $M(T/S)$ the fine moduli space of perverse point sheaves. Let $U \subset S$ be the maximal open set such that $f^{-1}\,|_{X_U}$ is an isomorphism. Denote by W the irreducible component of $M(T/S)$ which contains $U \subset S$.

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2. Fourier-Mukai type transforms on singular varieties

This section contains several basic lemmas on Fourier-Mukai type transforms. We essentially follow [5].

2.1 Boundedness of a transform

Let X and Y be quasi-projective varieties. Consider the diagram

$$\begin{array}{ccc} & X \times Y & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & Y. \end{array}$$

One can use the formula

$$\mathbf{R}p_{2\,*}(\mathcal{E} \overset{L}{\otimes} \mathbf{L}p_1^*(-))$$

to define a functor $F : D_{qc}(X) \to D_{qc}(Y)$ by results in [22]. When X and Y are smooth, every object $\mathcal{E} \in D^b(X \times Y)$ is of finite Tor-dimension. If $p_2|_{\mathrm{Supp}(\mathcal{E})} : \mathrm{Supp}(\mathcal{E}) \to Y$ is proper, then the functor F is also a functor of triangulated categories $F : D^b(X) \to D^b(Y)$.

However, an object $\mathcal{E} \in D^b(X \times Y)$ may not be of finite Tor-dimension when X and Y are not smooth. Hence this transform F may not send $D^b(X)$ to $D^b(Y)$. The next easy lemma shows that many such transforms $\mathbf{R}p_{2\,*}(\mathcal{E} \overset{L}{\otimes} \mathbf{L}p_1^*(-))$ send $D^b(X)$ to $D^b(Y)$.

Lemma 2.1. *Assume that $\mathcal{E} \in D^b(X \times Y)$ is isomorphic to a complex F of coherent $\mathcal{O}_{X\times Y}$-sheaves such that each of these sheaves is flat over $\mathcal{O}_X$, and* $\mathrm{Supp}(\mathcal{E}) \to Y$ *is a proper morphism. Then $\mathbf{R}p_{2\,*}(\mathcal{E} \overset{L}{\otimes} \mathbf{L}p_1^*(-))$ sends $D^b(X)$ to $D^b(Y)$.*

Proof. We first check the functor $\mathcal{E} \overset{L}{\otimes} \mathbf{L}p_1^*(-)$ sends $D^b(X)$ to $D^b(X \times Y)$. This can be checked locally and follows from the identity:

$$\left(M \overset{L}{\underset{C}{\otimes}} \left(N \overset{L}{\underset{A}{\otimes}} C\right)\right) \cong \left(M \overset{L}{\underset{A}{\otimes}} N\right),$$

where C is a ring flat over A and M is a finite complex of finitely presented C-modules and N is a finite complex of A-modules. Our assumption on Tor-dimension (of M over A) implies that $(M \overset{L}{\underset{A}{\otimes}} N)$ is a finite complex of finitely presented C-modules when N is a finite complex of finitely presented A-modules.

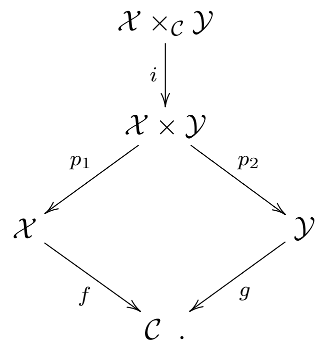
Let $\mathcal{F}$ be any object in $D^b(X)$. Write $\mathcal{G} = \mathcal{E} \overset{L}{\otimes} \mathbf{L}p_1^*(\mathcal{F}) \in D^b(X \times Y)$. Note that $\mathrm{Supp}(\mathcal{G}) \subset \mathrm{Supp}(\mathcal{E})$ is a closed subset, so $\mathbf{R}p_{2\,*}(\mathcal{G}) \in D^b(Y)$ by the assumption that $\mathrm{Supp}(\mathcal{E}) \to Y$ is proper. q.e.d.

Lemma 2.2. *Let Z be a closed subscheme of $X \times Y$ and $\mathcal{E}$ be an object in $D^b(Z)$. Denote by $i : Z \to X \times Y$ the inclusion map. Then we have*

$$\mathbf{R}p_{2\,*}\left(i_*\mathcal{E} \overset{L}{\otimes} \mathbf{L}p_1^*(-)\right) \cong \mathbf{R}(i \circ p_2)_*\left(\mathcal{E} \overset{L}{\otimes} \mathbf{L}(i \circ p_1)^*(-)\right).$$

Proof. This follows easily from the projection formula. q.e.d.

We use this lemma to prove the following fact. Consider the diagram



Let $\mathcal{E} \in D^b(\mathcal{X} \times \mathcal{Y})$ be an object which comes from $D^b(\mathcal{X} \times_{\mathcal{C}} \mathcal{Y})$. Then the Fourier-Mukai type transform $F_{\mathcal{E}}$ can be defined as

$$\mathbf{R}(p_2 \circ i)_* \left(\mathbf{L} \left((p_1 \circ i)^*(-) \overset{L}{\otimes} \mathcal{E} \right) \right)$$

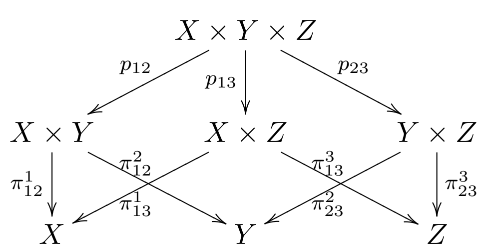
by the lemma.

2.2 Compositions of Fourier-Mukai type transforms

The next proposition shows that the composition of two Fourier-Mukai type transforms is still a Fourier-Mukai type transform. This is a generalization of Proposition 1.4 in [5].

Let X, Y and Z be quasi-projective varieties and I, J objects of $D^b(X \times Y)$ and $D^b(Y \times Z)$(resp.). We assume that I and J satisfy the assumptions in Lemma 2.1.

Consider the diagram of projections



and the functors

$$F_I : D^b(X) \to D^b(Y),$$

$$F_J : D^b(Y) \to D^b(Z),$$

defined by the formulas

$$F_I = \mathbf{R}\pi^2_{12\,*}(I \overset{L}{\otimes} \mathbf{L}\pi^{1\,*}_{12}(-)),$$

$$F_J = \mathbf{R}\pi^3_{23\,*}(J \overset{L}{\otimes} \mathbf{L}\pi^{2\,*}_{23}(-)).$$

Proposition 2.3. *The composition functor of F_I and F_J is isomorphic to F_K with*

$$K = \mathbf{R}p_{13\,*}(\mathbf{L}p^*_{23}J \overset{L}{\otimes} \mathbf{L}p^*_{12}I).$$

Proof. We follow the argument in [5]:

$$\begin{aligned}
F_J \circ F_I &= \mathbf{R}\pi^3_{23\,*}(J \overset{L}{\otimes} \mathbf{L}\pi^{2\,*}_{23}(\mathbf{R}\pi^2_{12\,*}(I \overset{L}{\otimes} \mathbf{L}\pi^{1\,*}_{12}(-)))) \\
&\cong \mathbf{R}\pi^3_{23\,*}(J \overset{L}{\otimes} \mathbf{R}p_{23\,*}(\mathbf{L}p^*_{12}(I \overset{L}{\otimes} \mathbf{L}\pi^{1\,*}_{12}(-)))) && (2.1)\\
&\cong \mathbf{R}\pi^3_{23\,*}\mathbf{R}p_{23\,*}(\mathbf{L}p^*_{23}J \overset{L}{\otimes}(\mathbf{L}p^*_{12}(I \overset{L}{\otimes} \mathbf{L}\pi^{1\,*}_{12}(-)))) && (2.2)\\
&\cong \mathbf{R}\pi^3_{13\,*}\mathbf{R}p_{13\,*}(\mathbf{L}p^*_{23}J \overset{L}{\otimes}(\mathbf{L}p^*_{12}(I) \overset{L}{\otimes} \mathbf{L}p^*_{12}\mathbf{L}\pi^{1\,*}_{12}(-))) && (2.3)\\
&\cong \mathbf{R}\pi^3_{13\,*}\mathbf{R}p_{13\,*}((\mathbf{L}p^*_{23}J \overset{L}{\otimes} \mathbf{L}p^*_{12}I) \overset{L}{\otimes} \mathbf{L}p^*_{13}\mathbf{L}\pi^{1\,*}_{13}(-)) && (2.4)\\
&\cong \mathbf{R}\pi^3_{13\,*}(\mathbf{R}p_{13\,*}(\mathbf{L}p^*_{23}J \overset{L}{\otimes} \mathbf{L}p^*_{12}I) \overset{L}{\otimes} \mathbf{L}\pi^{1\,*}_{13}(-)). && (2.5)
\end{aligned}$$

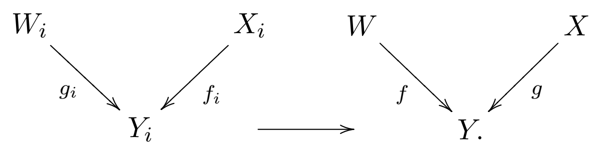
The isomorphism (2.1) follows from the flat base change theorem, the isomorphisms (2.2) and (2.5) follow from the projection formula. The isomorphisms (2.3) and (2.4) are obvious. q.e.d.

3. Reduction of the proof to affine cases

Let $f : X \to Y$ be a flopping contraction between two quasi-projective three-dimensional normal varieties. Assume that the variety X has only terminal Gorenstein singularities. We explain in this section how to reduce the proof of part (2) in Theorem 1.1 to an affine cover $\{Y_i\}$. Consider the diagram

$$\begin{array}{ccc}
W & & X \\
& {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle f} & \\
& Y. &
\end{array}$$

Fix an affine cover $\{Y_i\}$ of Y. Pull back everything to Y_i



Let $F : D^b(W) \to D^b(X)$ be a Fourier-Mukai type transform defined by an object $\mathcal{E} \in D^b(W \times X)$. Assume that $\mathcal{E}$ is of finite Tor-dimension over W and the projection morphism $W \times X \to X$ is proper when restricted to $\mathrm{Supp}(\mathcal{E}) \to X$.

Denote by F_i the corresponding Fourier-Mukai type transforms when we pull back everything to Y_i. Note that any Fourier-Mukai type transform also defines a functor on D_{qc}. We show that if we can check the equivalence of categories locally, then by the existence of a global adjoint functor, we are able to prove the equivalence of derived categories.

Remark 3.1. In the proof on Lemma 3.6, we need to work on D_{qc} since we invoke a theorem by Neeman on a very general form of Grothendieck duality (see [17]). Since this is the only reason for passing to D_{qc}, we would like to have a proof without using these huge categories. For the time being, however, we are not able to give such a proof.

Proposition 3.2 (= Proposition 1.2)**.** *With the notation as above, assume that all $F_i : D^b(W_i) \to D^b(X_i)$ are equivalences of derived categories. Then the Fourier-Mukai type transform $F : D^b(W) \to D^b(X)$ is an equivalence of derived categories.*

We give several lemmas needed for the proof in the subsequent subsections. The proof of this proposition is given at the end of this section.

3.1 A spanning class

We recall the definition of spanning classes for a triangulated category $\mathcal{A}$ (see Definition 2.1. in [7]).

Definition 3.3. A subclass Ω of objects of $\mathcal{A}$ is called a spanning class for $\mathcal{A}$, if for every object $a \in \mathcal{A}$

$$\mathrm{Hom}^i_{\mathcal{A}}(b, a) = 0 \ \ \forall\, b \in \Omega \ \ \forall i \in Z \Longrightarrow a \cong 0,$$

$$\mathrm{Hom}^i_{\mathcal{A}}(a, b) = 0 \ \ \forall\, b \in \Omega \ \ \forall i \in Z \Longrightarrow a \cong 0.$$

Lemma 3.4. *Let X be a normal projective variety with only isolated singular points $\{x_i : i = 1, \dots, k\}$. Let $\Omega_1 = \{\mathcal{O}_x : x \in X\}$ and $\Omega_2 = \{\mathcal{O}_Z : \operatorname{Supp}(Z) \subset X_{\mathrm{sing}} = \{x_i : i = 1, \dots, k\}\}$. Then $\Omega = \Omega_1 \bigcup \Omega_2$ is a spanning class for $D^b(X)$.*

Proof. (a) We check the condition

$$\operatorname{Hom}^i_{D^b(X)}(a, b) = 0 \ \ \forall\, b \in \Omega \ \ \forall i \in Z \Longrightarrow a \cong 0$$

by using the argument in [7]. For any object $a \in D^b(X)$ and any $x \in X$, there is a spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_X(H^{-q}(a), \mathcal{O}_x) \Rightarrow \operatorname{Hom}^{p+q}_{D^b(X)}(a, \mathcal{O}_x).$$

If a is nonzero, let q_0 be the maximal value of q such that H^q is nonzero. Take any point $x \in \operatorname{Supp}(a)$. There is a nonzero element of $E_2^{0,-q_0}$, which survives at the E_∞ stage. This gives an element of $\operatorname{Hom}_{D^b(X)}(a, \mathcal{O}_x)$, a contradiction.

(b) The condition

$$\operatorname{Hom}^i_{D^b(X)}(b, a) = 0 \ \ \forall\, b \in \Omega \ \ \forall i \in Z \Longrightarrow a \cong 0$$

is equivalent to the following statement:

$$a \not\cong 0 \Longrightarrow \operatorname{Hom}_{D^b(X)}(b, a) \neq 0 \text{ for some } b \in \Omega.$$

We use a similar spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p_X(b, H^q(a)) \Rightarrow \operatorname{Hom}^{p+q}_{D^b(X)}(b, a)$$

to prove this statement.

Fix any $x \in X_{\mathrm{reg}}$.

Claim 3.5. $\operatorname{Hom}^i_{D^b(X)}(\mathcal{O}_x, a) = 0 \ \ \forall\ i \ \Longrightarrow x \notin \operatorname{Supp}(a)$.

It is clear that for each i the sheaf $\mathcal{H}om^i_{D^b(X)}(\mathcal{O}_x, a)$ is a coherent $\mathcal{O}_x$-sheaf. Take an affine neighborhood $U = Spec(A)$ of x. There is no higher derived functor for $\Gamma(\text{Spec } (A)\ , -)$. Thus $\operatorname{Hom}^i_{D^b(X)} \cong \mathcal{H}om^i_A$. Since x is a nonsingular point, the sheaf $\mathcal{O}_x$ has a finite flat resolution. Thus $\mathbf{R}\operatorname{Hom}_{D^b(X)}(\mathcal{O}_x, a) \overset{L}{\otimes} \mathcal{O}_x \cong \mathbf{R}\operatorname{Hom}_{D^b(X)}(\mathcal{O}_x, a \overset{L}{\otimes} \mathcal{O}_x)$. By assumption we have

$$\mathbf{R}\operatorname{Hom}_{D^b(X)}(\mathcal{O}_x, a) = 0,$$

and hence

$$\mathbf{R}\mathrm{Hom}_{D^b(X)}(\mathcal{O}_x, a \overset{L}{\otimes} \mathcal{O}_x) = 0.$$

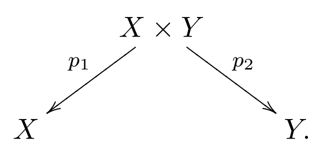
Replacing a by $a \overset{L}{\otimes} \mathcal{O}_x$, we may assume that a is with proper support.

Since X is projective, X_{reg} is quasi-projective. Both a and $\mathcal{O}_x$ are with proper supports. Serre duality implies that $\mathrm{Hom}^i_{D^b(X)}(a, \mathcal{O}_x) = 0$. By the argument in (a) above, it follows that $x \notin \mathrm{Supp}(a)$. This shows that $\mathrm{Supp}(a) \subset X_{\mathrm{sing}}$, which is equivalent to Claim 3.5.

Since $a \in D^b(X)$, there is a subscheme structure z on x_0 such that $a \in D^b(z)$ (i.e., every cohomology group is an $\mathcal{O}_z$-module). Let q_1 be the minimal value of q such that $H^q(a) \neq 0$. It is clear that $\mathbf{R}\mathrm{Hom}_{D^b(X)}(\mathcal{O}_z, H^{q_1}(a)) \neq 0$ and its elements survive at the E_∞ level. This concludes the proof. q.e.d.

3.2 Right adjoints

Lemma 3.6. *Let X and Y be projective Gorenstein varieties and $\mathcal{E}$ an object of $D_{qc}(X \times Y)$. Consider the diagram*



Denote by $F : D_{qc}(X) \to D_{qc}(Y)$ this Fourier-Mukai type transform. Then F has a right adjoint G.

Proof. We use the following isomorphisms:

$$\mathrm{Hom}_{D_{qc}(Y)}(\mathbf{R}p_{2\,*}\mathbf{L}p_1^*(A) \overset{L}{\otimes} \mathcal{E}, B)$$

$$\cong \mathrm{Hom}_{D_{qc}(X\times Y)}(\mathbf{L}p_1^*(A) \overset{L}{\otimes} \mathcal{E}, p_2^! B) \tag{3.1}$$

$$\cong \mathrm{Hom}_{D_{qc}(X\times Y)}(\mathbf{L}p_1^*(A), \mathbf{R}\mathcal{H}om(\mathcal{E}, p_2^! B)) \tag{3.2}$$

$$\cong \mathrm{Hom}_{D_{qc}(X)}(A, \mathbf{R}p_{1\,*}\, \mathbf{R}\mathcal{H}om(\mathcal{E}, p_2^! B)). \tag{3.3}$$

The isomorphism (3.1) follows from Grothendieck duality (see [17]). The isomorphism (3.2) follows from the fact that $(\otimes, \ \mathcal{H}om)$ is an adjoint pair. The last isomorphism is a consequence of the fact that $(\mathbf{L}p_1^*, \mathbf{R}p_{1\,*})$ is an adjoint pair. Thus F has a right adjoint G. q.e.d.

Remark 3.7. When the object $\mathcal{E}$ satisfies the assumptions in Lemma 2.1, we have $G(b) \in D^+(X)$ for all $b \in D^b(Y)$ by the explicit formula of the right adjoint G.

3.3 Conclusion of the proof

Proof of Proposition 3.2. By Lemma 3.6, the Fourier-Mukai type transform $F : D_{qc}(W) \to D_{qc}(X)$ has a right adjoint $G : D_{qc}(X) \to D_{qc}(W)$, so we have the natural transforms $\mathrm{id}_{D_{qc}(W)} \to GF$ and $FG \to id_{D_{qc}(X)}$. To show $F : D^b(W) \to D^b(X)$ is an equivalence, it suffices to show that $a \cong GF(a)$ for all $a \in D^b(W)$ and $FG(b) \cong b$ for all $b \in D^b(X)$.

For each $a \in D^b(W)$ we have a distinguished triangle in $D_{qc}(W)$

$$(*) \qquad \to a \to GF(a) \to c \to a[1] \to .$$

To show that $a \cong GF(a)$, it amounts to showing $c \cong 0$. We first show a weaker claim.

Claim 3.8. $c \in D^b(W)$.

Note that Claim 3.8 is equivalent to the fact that $GF(a) \in D^b(W)$. Pulling back everything to each Y_i, we get a distinguished triangle in $D_{qc}(W_i)$

$$(*)_i \qquad \to a_i \to G_iF_i(a_i) \to c_i \to a_i[1] \to$$

for each Y_i.

Note that for every $x \in D^b(X)$ we have $G(x) \in D^+(W)$ by the explicit formula of the right adjoint functor, so $GF(c) \in D^+(W)$ (see Remark 3.7).

Since $F_i : D^b(W_i) \to D^b(X_i)$ is an equivalence by assumption, it follows that $\mathrm{Hom}^j_{D_{qc}(W_i)}(x_i, c_i) = 0$ for all j and all $x_i \in D^b(W_i)$. In fact, we only need F_i to be fully faithful for this assertion. To show $c_i \cong 0$, we use the following triangle for each k

$$\to \tau_{\leq k} c_i \to c_i \to \tau_{\geq k+1} c_i \to \tau_{\leq k} c_i[1] \to .$$

Since $c_i \in D^+(W_i)$, it follows that $\tau_{\leq k} c_i \in D^b(W_i)$ for all k. Taking $\mathrm{Hom}(\tau_{\leq k} c_i, -)$ into the above triangle, and noticing that

$$\mathrm{Hom}^0_{D_{qc}(W_i)}(\tau_{\leq k} c_i, \tau_{\geq k+1} c_i) = 0$$

and

$$\mathrm{Hom}^{-1}_{D_{qc}(W_i)}(\tau_{\leq k} c_i, \tau_{\geq k+1} c_i) = 0,$$

it follows that

$$\mathrm{Hom}^0_{D_{qc}(W_i)}(\tau_{\le k}c_i, \tau_{\le k}c_i) \cong \mathrm{Hom}^0_{D_{qc}(W_i)}(\tau_{\le k}c_i, c_i),$$

which is 0 since $\tau_{\le k}c_i \in D^b(W_i)$.

If $c_i \not\cong 0$, then we can choose a k such that $\tau_{\le k}c_i \not\cong 0$. For such a k, we have $\mathrm{Hom}^0_{D_{qc}(W_i)}(\tau_{\le k}c_i, \tau_{\le k}c_i) \neq 0$, a contradiction. This shows that $c_i \cong 0$. In particular, $c_i \in D^b(W_i)$, so $c \in D^b(W)$. This proves Claim 3.8.

Let Ω be as in Lemma 3.4. Let $y \in \Omega$. Taking $\mathrm{Hom}(y, -)$ into the distinguished triangle $(*)$ and the distinguished triangle $(*)_i$ for each Y_i, we get the following exact sequences

$$\begin{array}{ccccccccc}
\mathrm{Hom}^j(y,a) & \longrightarrow & \mathrm{Hom}^j(y,GF(a)) & \longrightarrow & \mathrm{Hom}^j(y,c) & \longrightarrow & \mathrm{Hom}^{j+1}(y,a) & \longrightarrow & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}^j(y_i,a_i) & \longrightarrow & \mathrm{Hom}^j(y_i,G_iF_i(a_i)) & \longrightarrow & \mathrm{Hom}^j(y_i,c_i) & \longrightarrow & \mathrm{Hom}^{j+1}(y_i,a_i) & \longrightarrow & .
\end{array}$$

Note that the support of y lies in some Y_i since $y \in \Omega$. Fix such a scheme Y_i. We have $y_i \cong y$ and all vertical arrows are isomorphisms. Since (F, G) and (F_i, G_i) are adjoint pairs, it follows that

$$\mathrm{Hom}^j_{D_{qc}(W)}(y, GF(a)) \cong \mathrm{Hom}^j_{D_{qc}(X)}(F(y), F(a))$$

and

$$\mathrm{Hom}^j_{D_{qc}(W_i)}(y_i, G_iF_i(a_i)) \cong \mathrm{Hom}^j_{D_{qc}(X_i)}(F_i(y_i), F_i(a_i)).$$

Together with our assumption that all F_i are equivalences, this implies $\mathrm{Hom}^j_{D_{qc}(W)}(y, c) = 0$ for all $y \in \Omega$. Since $c \in D^b(W)$ and Ω is a spanning class for $D^b(W)$, it follows that $c \cong 0$.

To show $FG(b) \cong b$ for all $b \in D^b(X)$, we need to use the assumption that F_i is an equivalence. Note that from Claim 3.8 we have $G_iF_i(a_i) \in D^b(W_i)$ for all $a_i \in D^b(W_i)$, and since $F_i : D^b(W_i) \to D^b(X_i)$ is an equivalence it follows that $G_i : D^b(X_i) \to D^b(W_i)$. Therefore (F_i, G_i) is also an adjoint pair when we work on D^b, so $G_i : D^b(X_i) \to D^b(W_i)$ is also an equivalence. Using another distinguished triangle

$$\to FG(b) \to b \to c \to FG(b)[1] \to,$$

one can show that $FG(b) \cong b$ by a similar argument. This concludes the proof. q.e.d.

4. Basic properties of $M(X/Y)$ and $W(X/Y)$

We prove some basic lemmas on the distinguished component W of $M(X/Y)$.

4.1 Characterization of the universal perverse ideal sheaf

We begin with the next lemma.

Lemma 4.1. *Let S and T be two integral schemes and F be a coherent sheaf on $S \times T$. Let π be the projection map $S \times T \to T$. Assume the following two conditions:*

(1) *F is flat over S.*

(2) *There is a dense open set $U \subset S$ such that F is torsion free on $\pi^{-1}(U)$.*

Then the sheaf F is torsion free.

Proof. The problem is local, so we may assume that both S and T are affine schemes. We use torsion sections to get a contradiction.

Assume that F is not torsion free. Let x be a torsion section. Denote by $V(y)$ the zero scheme of y for a regular function y on S. By the assumption (2), the image of the support of x under the projection, denoted by $\pi_S(\mathrm{Supp}(x))$, is a proper subscheme of S. We can find a regular function s on S such that $\pi_S(\mathrm{Supp}(x)) \subset V(s)$ and the regular function s annihilates x (we consider s as a regular function on $S \times T$ by the natural map of rings induced by the projection map).

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{S\times T} \xrightarrow{\cdot\, s} \mathcal{O}_{S\times T} \longrightarrow \mathcal{O}_{V(s)\times T} \longrightarrow 0.$$

Tensoring this with F, we get a right exact sequence

$$F \longrightarrow F \longrightarrow F \otimes \mathcal{O}_{V(s)\times T} \longrightarrow 0.$$

The map on the left is the multiplication by s. Since $xs = 0$, it is not injective. This shows that $\mathrm{Tor}_1(F, \mathcal{O}_{V(s)\times T}) \neq 0$. This implies that F is not flat, a contradiction. q.e.d.

We give a proposition on the universal ideal sheaf.

Proposition 4.2. *The universal perverse ideal sheaf is the ideal sheaf $\mathcal{I}_{W\times_Y X}$ of the fiber product, consequently the universal perverse point sheaf is $\mathcal{O}_{W\times_Y X}$.*

Proof. Let F be the universal ideal sheaf and $\alpha : F \to \mathcal{O}_{W\times X}$ be the corresponding homomorphism between sheaves. Denote by Γ the graph of $g : W \to Y$. The sheaf F is flat over W by definition. It is clear that F is torsion free on the dense open set $U \times X$, where U is the isomorphic locus of $f : X \to Y$ and is considered as an open set inside both X, Y and W.

By Lemma 4.1, it follows that F is indeed torsion free. Since the morphism $\alpha : F \to \mathcal{O}_{W\times X}$ is generically injective, the kernel is a torsion subsheaf. By Lemma 4.1 again, it follows that the homomorphism α is injective. So we can identify F as an ideal sheaf of $\mathcal{O}_{W\times X}$.

We show that $F = \mathcal{I}_{W\times_Y X}$. As shown in [8], we have that $f_*(F) = I_\Gamma$, the ideal sheaf of the graph in $W \times Y$. By Proposition 5.1 in [8] , it follows that the natural map $f^*f_*(F) \to F$ is surjective.

Since $f_*(F) = I_\Gamma$, the images of $f^*f_*(F)$ and $f^*f_*(\mathcal{I}_{W\times_Y X})$ in $\mathcal{O}_{W\times X}$ coincide. This shows that $F = \mathcal{I}_{W\times_Y X}$. q.e.d.

4.2 Flatness lemma

Proposition 4.3. *Let X_1 be an irreducible quasi-projective variety. Consider the diagram:*

$$\begin{array}{ccc} X_1 \times_Y X & \longrightarrow & X \\ \downarrow & & \downarrow \\ X_1 & \xrightarrow{\ f_1\ } & Y. \end{array}$$

If the ideal $\mathcal{I}_{X_1\times_Y X}$ in $\mathcal{O}_{X_1\times X}$ is flat over $\mathcal{O}_{X_1}$ and the image of X_1 is not contained in the image of the exceptional set of X/Y in Y, then there is a canonical morphism $h : X_1 \to W(X/Y)$.

Proof. Let U be the isomorphic locus of $X \to Y$. We consider U as an open subset both in X and Y. Pick a point $x_1 \in X_1$ such that $u = f_1(x_1) \in U$. The sheaf $\mathcal{I}_{X_1\times_Y X,\, x_1}$ is $\mathcal{I}_u$, the ideal of the point $u \in U$. Since $\mathcal{I}_{X_1\times_Y X}$ is flat, this family of sheaves has the correct numerical class, say γ.

The scheme $W(X/Y)$ is isomorphic to $M_{PI}(X/Y, \gamma)$, the moduli space of perverse ideal sheaves with the numerical equivalence class γ. It suffices to show that there is a morphism $h : X_1 \to M_{PI}(X/Y, \gamma)$, which amounts to showing that $\mathcal{I}_{X_1\times_Y X}$ is a family of perverse ideal sheaves.

This would follow if we can show that the natural homomorphisms

$$f^* f_*(\mathcal{I}_{X_1 \times_Y X,\, x_1}) \to \mathcal{I}_{X_1 \times_Y X,\, x_1}$$

are surjections for all $x_1 \in X_1$. This holds if the natural homomorphism

$$f_{X_1}^* f_{X_1\,*}(\mathcal{I}_{X_1 \times_Y X}) \to \mathcal{I}_{X_1 \times_Y X}$$

is a surjection, which follows since $f_{X_1\,*}(\mathcal{I}_{X_1 \times_Y X})$ is the ideal of the graph $f_1 : X_1 \to Y$. q.e.d.

4.3 Relations between $W(X/Y)_T$ and $W(S/T)$

Let $X \to Y$ be a flopping contraction between three-dimensional normal varieties. Assume that X has at worst terminal Gorenstein singularities. By standard results on flops, the variety Y has at worst terminal Gorenstein singularities (see Theorem 6.14 in [16]). Let $T \subset Y$ be an effective Cartier divisor; for simplicity we assume that it is an integral subscheme of Y. Consider the diagram

$$\begin{array}{ccc} S & \xrightarrow{i_S} & X \\ {\scriptstyle f_T}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{i_T} & Y. \end{array}$$

Note that the conditions (B.1) and (B.2) hold for the morphism $S \to T$.

The condition (B.2) is clear. We now show the condition (B.1). It is clear that $f_{T\,*}(\mathcal{O}_S) = \mathcal{O}_T$. To show $\mathbf{R}^i f_T(\mathcal{O}_S) = 0$ for all $i \geq 1$, we apply the theorem on formal functions (see p.277 in [11]). It suffices to show that

$$H^i(S_t, \mathcal{O}_t) = 0 \;\; \forall\, i \geq 1$$

for all $t \in T$. Since

$$\mathbf{R}^i f(\mathcal{O}_X) = 0 \;\; \forall\, i \geq 1$$

by assumption, it follows that

$$H^i(X_y, \mathcal{O}_y) = 0 \;\; \forall\, i \geq 1$$

for all $y \in Y$. For any $t \in T \subset Y$ the fibers X_t and S_t are canonically isomorphic to each other. This implies that

$$\mathbf{R}^i f_T(\mathcal{O}_S) = 0 \ \ \forall\, i \geq 1.$$

Thus the condition (B.1) also holds for $S \to T$.

Proposition 4.4 (= Proposition 1.6)**.** *There is a natural embedding $W(X/Y)_T \hookrightarrow M(S/T)$, which is an inclusion of components.*

Proof. (a) We show that there is a canonical morphism $M(S/T) \to M(X/Y)_T$.

Let $p \in M(S/T)$. Denote the corresponding perverse point sheaf for $S \to T$ by E_p. It is clear that if for a point $p \in M(S/T)$, the corresponding object E_p is also a perverse point sheaf for $X \to Y$, then this point, which we still denote by p, must lie in the fiber $M(X/Y)_T$. Let

$$0 \to I_{E_p} \to \mathcal{O}_S \to E_p \to 0$$

be the exact sequence in the abelian category Per (S/T).

Step 1. We show that for every point $p \in M(S/T)$, the corresponding perverse point sheaf E_p for $S \to T$ is a perverse sheaf for $X \to Y$.

This follows easily by checking the conditions (PS.1)-(PS.3) of Lemma A.2.

Step 2. We show that I_{E_p} is also a perverse sheaf.

This again follows by checking the conditions (PS.1)-(PS.3).

Combining results from Step 1 and Step 2, it follows that

$$0 \to I_{E_p} \to \mathcal{O}_S \to E_p \to 0$$

is also an exact sequence in the abelian category Per (X/Y).

Step 3. The sheaf $\mathcal{O}_S$ is a perverse structure sheaf for $X \to Y$.

Consider the exact sequence of sheaves

$$0 \to I_S \to \mathcal{O}_X \to \mathcal{O}_S \to 0.$$

It suffices to check that I_S is a perverse ideal sheaf. This follows since the conditions (PIS.1) and (PIS.2) of Proposition A.5 are satisfied.

Composing two surjections $\mathcal{O}_X \to \mathcal{O}_S$ and $\mathcal{O}_S \to E_p$, we obtain the surjection $\mathcal{O}_X \to E_p$ in the abelian category Per(X/Y). This shows that

E_p is a perverse point sheaf for $X \to Y$. Since $M(X/Y)$ is a fine moduli space, we have an embedding $M(S/T) \to M(X/Y)_T$.

(b) We show that there is an embedding $W(X/Y)_T \to M(S/T)$.

For each point $w \in W(X/Y)_T$, let E_w be the corresponding perverse point sheaf.

Step 1. We prove that E_w is a perverse sheaf for $S \to T$.

The main point is that E_w is indeed a complex of $\mathcal{O}_S$-modules since the universal perverse point sheaf is the structure sheaf of the fiber product $W \times_Y X$. By checking the conditions (PS.1)-(PS.3) of Lemma A.2 in Appendix A, it follows that $E_w \in \mathrm{Per}(S/T)$.

Step 2. The sheaf $\mathcal{O}_S$ is a perverse structure sheaf for $X \to Y$.

This is proven in part (a).

Step 3. The sheaf E_w is a perverse point sheaf for $S \to T$.

The morphism $\mathcal{O}_X \to E_w$ factors through $\mathcal{O}_S$. By Step 2 the morphism $\mathcal{O}_X \to \mathcal{O}_S$ is a surjection in the abelian category $\mathrm{Per}(X/Y)$, so $\mathcal{O}_S \to E_w$ is also a surjection by standard results on abelian categories, which shows the corresponding kernel, denoted by I_{E_w}, is also a perverse sheaf. Note that the object I_{E_w} is also a shifting of the cone of $\mathcal{O}_S \to E_w$, from which follows that I_{E_w} is a complex of $\mathcal{O}_S$-modules. Abusing the notation, we denote by I_{E_w} and E_w the objects in $\mathrm{Per}(S/T)$ such that $I_{E_w} \to \mathcal{O}_S \to E_w$ is a distinguished triangle in $D^b(S)$ and the push-forward of this triangle is the exact sequence

$$0 \to I_{E_w} \to \mathcal{O}_S \to E_w \to 0$$

in the abelian category $\mathrm{Per}(X/Y)$. Since E_w is in the correct numerical class, it follows that E_w is a perverse point sheaf for $S \to T$. This gives an embedding $W(X/Y)_T \hookrightarrow W(S/T)$ since $W(S/T)$ is a fine moduli space.

Combining the results in (a) and (b), we obtain two morphisms $M(S/T) \to M(X/Y)_T$, and $W(X/Y)_T \to M(S/T)$. Each of these two morphisms is an embedding. By our construction, the composition $W(X/Y)_T \to M(S/T) \to M(X/Y)_T$ is an inclusion of components, which implies that each morphism is an inclusion of components. This concludes the proof. q.e.d.

Since $W(X/Y)$ is a fine moduli space, pulling back the universal object over $W(X/Y)$ via the canonical embedding $W(S/T) \to W(X/Y)$ in part (a), one obtains the following corollary of Proposition 4.4.

Corollary 4.5. *The universal perverse point sheaf for the morphism* $S \to T$ *is* $\mathcal{O}_{W(S/T)\times_T S} \cong \mathbf{L}i^*(\mathcal{O}_{W(X/Y)\times_Y X})$.

5. Deformations and general hyperplane sections

The proof in this section was inspired by helpful discussions with M. Reid. Throughout this section we assume that Y is an affine variety. Let $f : X \to Y$ be a crepant projective birational morphism between two quasi-projective three-dimensional normal Gorenstein varieties. Assume that the variety X has at worst terminal singularities. Denote the exceptional set by C. Under these assumptions, all singularities of X are isolated hypersurface singularities (see [16] p. 169). By standard results in the MMP, it is well-known that Y is also terminal (see Theorem 6.14 in [16]).

We first show that for a general one-parameter deformation $F : \mathcal{X} \to \mathcal{Y}$ of $f : X \to Y$ the total space $\mathcal{X}$ is nonsingular. Then we show that the hyperplane section $\mathcal{S}$, the preimage of a general member $\mathcal{T}$ of a suitable linear system of divisors passing through the singular points $Y_{\text{sing}} = \{p_i : i = 1, \ldots, m\} \subset Y$, is nonsingular. In the first part we use the fact that these singularities are hypersurface singularities. The second part can be reduced to showing that the preimage of a general hyperplane passing through $Y_{\text{sing}} \in Y$ has only canonical singularities.

Let $V_0 \subset H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ be any linear subsystem of divisors passing through $Y_{\text{sing}} = \{p_i : i = 1, \ldots, m\}$ such that $|BsV_0| = Y_{\text{sing}}$ (as a scheme). Let T be a general element of V_0. Denote the preimage of T in X by S.

Proposition 5.1. *The preimage* S *of a general element* T *of* $V_0 \subset H^0(Y, \mathcal{O}_Y) = H^0(X, \mathcal{O}_X)$ *has only canonical singularities.*

Proof. First note that S is a Gorenstein variety since it is a hyperplane section of a Gorenstein variety X. The divisor T has only canonical singularities. By a Bertini type theorem, the Cartier divisor T is nonsingular outside Y_{sing}. Therefore S is nonsingular outside the exceptional curves C.

We show that S is normal and has only canonical (Du Val) singularities. We have $K_S = f^*K_T$ by:

(1) $K_S = K_X|_S + S|_S$ and $K_T = K_Y|_T + T|_T$ (by the adjunction formula).

(2) $K_X = f^*K_Y$.

Consider the normalization $g : S^n \to S$. We have $\omega_{S^n} = (\mathcal{C})g^*(\omega_S)$, where $\mathcal{C}$ is the conductor ideal. Since T has only Du Val singularities, we have $(g \circ f)^*(\omega_T) \subset \omega_{S^n}$. This shows that S is normal. To complete the proof, we compute the discrepancies. Take a resolution $h : V \to S$ of S. We have

$$K_V = h^* K_S + \sum a_i E_i = (h \circ f)^* K_T + \sum a_i E_i$$

where E_i's are the exceptional divisors. Since T has only canonical singularities, it follows that S has only canonical singularities. q.e.d.

Proposition 5.2 (= Proposition 1.8)**.** *A general one-parameter deformation of $f : X \to Y$ is nonsingular.*

Proof. Let $\mathcal{X}_{\text{univ}}$ be the semiuniversal object over the semiuniversal deformation space $\text{Def}(X)$. Let $\mathcal{Y} = \text{Spec}(\mathcal{O}_{\mathcal{X}_{\text{univ}}})$. Then $\mathcal{Y}$ is a deformation of Y, and hence the natural morphism $F : \mathcal{X}_{\text{univ}} \to \mathcal{Y}$ is a deformation of $f : X \to Y$. Thus it suffices to deform a Zariski neighborhood of $f^{-1}(p)$ in X.

Since X has only isolated hypersurface singularities, the deformation space of X is $\text{Ext}^1(\Omega_X, \mathcal{O}_X)$. We show below that the obstruction group $\text{Ext}^2(\Omega_X, \mathcal{O}_X) = 0$. To compute $\text{Ext}^2(\Omega_X, \mathcal{O}_X)$, we use the following spectral sequence

$$H^p(X, \mathcal{E}xt^q(\Omega_X, \mathcal{O}_X)) \Rightarrow \text{Ext}^{p+q}(\Omega_X, \mathcal{O}_X).$$

Since X has only isolated hypersurface singularities, it is clear that $\mathcal{E}xt^2(\Omega_X, \mathcal{O}_X)$ is 0. We also know that $H^1(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = 0$ since $\text{Supp}(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))$ is isolated. It remains to show that

$$H^2(X, \mathcal{E}xt^0(\Omega_X, \mathcal{O}_X)) = 0.$$

This follows from the Leray spectral sequence

$$H^p(Y, \mathbf{R}^q f_*(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F})$$

and $H^i(\text{Spec}\,(A), \mathcal{F}) = 0$ for $i \geq 1$. By a similar argument, one could obtain that $E_2^{p,q} = 0$ for $p + q \geq 2$, though we do not need this more general fact in our proof.

Since every $E_2^{p,q} = 0$ for $p + q = 2$, we get the following short exact sequence

$$0 \to H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to \text{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to 0.$$

The important point is that the map

$$\mathrm{Ext}^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))$$

is surjective. Thus every deformation of the singularity can be lifted to a deformation of X. Since X has finitely many singularities and that smoothness at a given point is an open condition, it suffices to check the smoothness statement in neighborhoods of each singular point of X.

Note that we can check whether a variety is nonsingular at a given point x locally analytically. Thus we shall work locally analytically in the remainder of this argument. Denote the semiuniversal deformation space of the singularity $x \in X$ by $\mathrm{Def}\,(x \in X)$ and the semiuniversal object over $\mathrm{Def}\,(x \in X)$ by $\mathcal{X}$. For an isolated hypersurface singularity, the total space $\mathcal{X}$ over the semiuniversal deformation space $\mathrm{Def}\,(x \in X)$ is nonsingular by the explicit description of the semiuniversal space and the total space.

The variety $\mathcal{X}$ is analytically isomorphic to $f(x, y, z, w) + t_1 f_1 + \cdots + t_n f_n = 0$ where n is the dimension of $\mathrm{Def}(x \in X)$ and f_i are suitable polynomials such that at least one of the f_i, say f_1, is nonzero at $(0, 0, 0, 0)$.

The canonical morphism $\{0 \in \mathrm{Def}(X)\} \to \{0 \in \mathrm{Def}(x \in X)\}$ gives a linear map on tangent spaces. This map is surjective. We write the defining equation for the semiuniversal object $\mathcal{X}_{\mathrm{univ}}$ as $F = f(x, y, z, w) + t_1 g_1 + \cdots + t_m g_m = 0$. Choose a direction of $c = (c_1, ..., c_m)$ in the tangent space of $\mathrm{Def}(X)$ such that its image under the induced linear map is $(1, 0, \ldots, 0)$. This gives a smoothing of the singular point x by the Jacobi criterion. q.e.d.

Let $F : \mathcal{X} \to \mathcal{Y}$ be a one-parameter deformation of $f : X \to Y$ such that $\mathcal{X}$ is smooth. Let $\mathcal{T}$ be a general hyperplane section passing through $Y_{\mathrm{sing}} \in Y$. Denote by $\mathcal{S}$ the preimage of $\mathcal{T}$.

Proposition 5.3. *There exists a finite-dimensional vector space $V \subset H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ such that the preimage $\mathcal{S}$ of a general hyperplane section passing through p is nonsingular.*

Proof. Fix any V_0 satisfying the conditions at the beginning of this section. Let V be a finite-dimensional linear subsystem $V \subset H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. Denote by $\mathrm{Im}(V)$ the image of V in $H^0(Y, \mathcal{O}_Y)$ under the natural homomorphism $H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to H^0(Y, \mathcal{O}_Y)$. We choose a finite-dimensional linear subsystem $V \subset H^0(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ such that $\mathrm{Im}(V) \subset H^0(Y, \mathcal{O}_Y)$ contains V_0 and $|BsV| \cap Y = Y_{\mathrm{sing}} = \{p_i : i = 1, \ldots, m\}$. A general hyperplane section S of V_0 has only canonical surface singularities and hence has only hypersurface singularities. The subset of this linear system V such that the corresponding members are nonsingular at a specific point is an open set. There are only finitely many singular points on S. Combining these two facts, it suffices to check the corresponding open set is nonempty for each singular point. We divide the singular points of S into two types.

A point $x \in S_{\text{sing}}$ is called of type 1 if $x \in S_{\text{sing}} \bigcap X_{\text{sing}}$. A point $y \in S_{\text{sing}}$ is called of type 2 if $y \in S_{\text{sing}}/X_{\text{sing}}$.

We now show that every section $s \in H^0(X, \mathcal{O}_X)$ can be lifted to a section in $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. This follows easily from the exact sequence

$$0 \to H^0(\mathcal{X}, \mathcal{O}(-X)) \to H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to H^0(X, \mathcal{O}_X) \to H^1(\mathcal{X}, \mathcal{O}(-X))$$

and the fact that $H^1(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-X)) = 0$ (by the Leray spectral sequence and the fact that $\mathcal{Y}$ has only rational singularities). We still denote a lifting of s by s.

The variety $S \subset \mathcal{X}$ is a complete intersection. Denote the ideal by $\mathcal{I}_S = (s, g) \subset \mathcal{O}_{\mathcal{X}}$.

For a singular point x of type 1, we show that the divisor defined by g is nonsingular near x. We prove this by computing the embedding dimension of X at x. Passing to a formal or analytic neighborhood of $x \in \mathcal{X}$, we may assume that the ring of this formal neighborhood is $k[[x, y, z, w]]$. We have $m_{x,\,S}/m^2_{x,\,S} = (x, y, z, w)/(m^2_{x,\,S}, s, g)$. This vector space is of dimension 3 since S has a canonical surface singularity at x. Since x is a singular point of $X = \{s = 0\}$, it follows that $s \subset m^2_{x,\,S}$, which implies $\{g = 0\}$ is nonsingular at x.

For a singular point y of type 2 in S, the defining equation s of X is nonsingular at y. For a small enough ϵ the hyperplane section defined by $\epsilon \cdot g + s$ gives a divisor, which is nonsingular at y. q.e.d.

6. Equivalences of derived categories: dimension 4 to dimension 3

The proof in this section is based on suggestions of T. Bridgeland. We again assume that Y is an affine quasi-projective variety throughout this section. Let X be a quasi-projective threefold with only terminal Gorenstein singularities and $f : X \to Y$ be a flopping contraction. Let $W = W(X/Y)$ be the distinguished component of the moduli space of perverse point sheaves $M(X/Y)$. We prove in Section 5 that there is a deformation $F : \mathcal{X} \to \mathcal{Y}$ of $f : X \to Y$ with smooth $\mathcal{X}$. We summarize what we know:

(1) The Fourier-Mukai type transform $\Psi : D^b(\mathcal{W}) \to D^b(\mathcal{X})$ defined by the universal perverse point sheaf is an equivalence.

(2) $\mathcal{W}$ is smooth and is a flop of $\mathcal{X} \to \mathcal{Y}$.

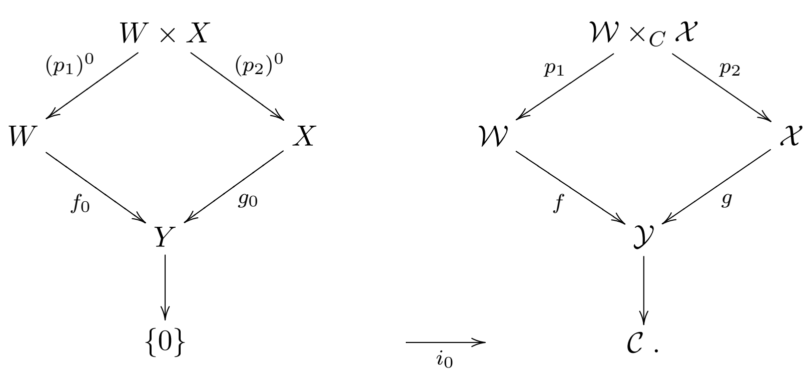
(3) $W \cong \mathcal{W}_Y$.

From what we know, it is a standard argument to deduce that $W(X/Y) \to Y$ is the flop of $X \to Y$. We sketch the argument here for the reader's convenience.

Since W is Gorenstein and generically reduced, it is a reduced scheme and hence is an integral scheme. Using the argument given in Proposition 5.1, it follows that W is normal and has at worst terminal singularities. By the adjunction formula, we have $K_W \cdot \mathcal{C} = K_{\mathcal{W}} \cdot \mathcal{C}$ for every curve $\mathcal{C} \subset W \subset \mathcal{W}$, which implies the canonical bundle K_W is g-trivial for $g : W \to Y$ since $K_{\mathcal{W}}$ is G-trivial. Let $\mathcal{D}_2 \subset \mathcal{W}$ be the effective divisor such that $-\mathcal{D}_2$ is G-ample and its birational transform $\mathcal{D}_1$ in $\mathcal{X}$ is F-ample. Intersect $\mathcal{D}_1$ with X, and denote the intersection by D_1. Similarly, let $D_2 = \mathcal{D}_2 \cap Y$. Then $-D_2$ is a g-ample divisor and D_1 is an f-ample divisor. To show W is a flop, it remains to show that the morphism g is not a divisorial contraction, which is evident since $K_W = g^*K_Y$ and Y has only terminal singularities.

Our goal in this section is to prove that $\Psi_0 : D^b(W) \to D^b(X)$ is an equivalence of categories (see below for the notation Ψ_0).

Consider the diagram



Denote by $\Psi_0 : D^b(W) \to D^b(X)$ the Fourier-Mukai type transform defined by the kernel $\mathbf{L}i_0^*(\mathcal{O}_{\mathcal{W}\times_{\mathcal{Y}}\mathcal{X}})$, which is equivalent to the

Fourier-Mukai type transform defined by the universal perverse point sheaf $\mathcal{O}_{W\times_Y X}$ for $X \to Y$ (see Corollary 4.5).

We claim that $\Psi(i_{0\,*}\mathcal{F}) \cong i_{0\,*} \circ \Psi_0(\mathcal{F})$ for $\mathcal{F} \in D^b(W)$. In fact, we prove a stronger lemma below.

Let $\mathcal{E} \in D^b(\mathcal{W} \times_{\mathcal{C}} \mathcal{X})$ be an object satisfying the assumptions in Lemma 2.1. We may also consider it as an object in $D^b(\mathcal{W} \times \mathcal{X})$. Let $F : D^b(\mathcal{W}) \to D^b(\mathcal{X})$ be the Fourier-Mukai type transform defined by $\mathcal{E}$. By Lemma 2.2, the functor F can be defined as $\mathbf{R}p_{1\,*}(\mathbf{L}p_2^*((-) \overset{L}{\otimes} \mathcal{E}))$. Denote by $F_0 : D^b(W) \to D^b(X)$ the Fourier-Mukai transform defined by the object $\mathbf{L}i_0^*\mathcal{E} \in D^b(W \times X)$.

Lemma 6.1 (= Proposition 1.9)**.** *Notation as above. Denote by F the Fourier-Mukai type transform defined by the object $\mathcal{E}$. Then $F(i_{0\,*}(-)) \cong i_{0\,*} \circ F_0(-)$.*

Proof. We use the following isomorphisms:

$$\begin{aligned} F(i_{0\,*}(-)) &= \mathbf{R}p_{1\,*}(\mathbf{L}p_2^*(i_{0\,*}(-) \overset{L}{\otimes} \mathcal{E})) \\ &\cong \mathbf{R}p_{2\,*}(\mathbf{R}i_{0\,*}(\mathbf{L}(p_1)^{0\,*}(-)) \overset{L}{\otimes} \mathcal{E}) \qquad (6.1)\\ &\cong \mathbf{R}p_{2\,*}(\mathbf{R}i_{0\,*}(\mathbf{L}(p_1)^{0\,*}(-) \overset{L}{\otimes} \mathbf{L}i_0^*\mathcal{E})) \qquad (6.2)\\ &\cong \mathbf{R}i_{0\,*}(\mathbf{R}(p_{2\,*})^0(\mathbf{L}(p_1)^{0\,*}(-) \overset{L}{\otimes} \mathbf{L}i_0^*\mathcal{E})). \qquad (6.3)\end{aligned}$$

The isomorphism (6.1) follows from the flat base change theorem. The isomorphism (6.2) follows from the projection formula. The isomorphism (6.3) is obvious. The last line is, by definition, the functor $i_{0\,*} \circ F_0(-)$. q.e.d.

Proposition 6.2 (=Proposition 1.10)**.** *Notation as above. Then*

$$\Psi : D^b(\mathcal{W}) \cong D^b(\mathcal{X}) \implies \Psi_0 : D^b(W) \cong D^b(X).$$

Proof. Applying Lemma 6.1 to Ψ, it follows that $\Psi(i_{0\,*}(-)) \cong i_{0\,*}(\Psi_0(-))$. Let $\Phi : D^b(\mathcal{X}) \to D^b(\mathcal{W})$ be the right adjoint functor of $\Psi : D^b(\mathcal{W}) \to D^b(\mathcal{X})$, and $\mathcal{E}_1$ be the object corresponding to the Fourier-Mukai type transform Φ.

Denote by $\Phi_0 : D^b(X) \to D^b(W)$ the Fourier-Mukai type transform defined by the object $\mathbf{L}i_0^*\mathcal{E}_1$. Lemma 6.1 also implies $\Phi(i_{0\,*}(-)) \cong i_{0\,*}(\Phi_0(-))$. These two facts give the following commutative diagram

$$\begin{array}{ccc} D^b(W) & \xrightarrow[i_0]{} & D^b(\mathcal{W}) \\ {\scriptstyle \Psi_0}\downarrow & & \downarrow{\scriptstyle \Psi} \\ D^b(X) & \xrightarrow[i_0]{} & D^b(\mathcal{X}) \\ {\scriptstyle \Phi_0}\downarrow & & \downarrow{\scriptstyle \Phi} \\ D^b(W) & \xrightarrow[i_0]{} & D^b(\mathcal{W}). \end{array}$$

Combining the top and the bottom parts of this diagram, it follows that $\Phi \circ \Psi(i_0\,(-)) \cong i_{0\,*}(\Phi_0 \circ \Psi_0(-))$. The functor $\Phi \circ \Psi$ is the Fourier-Mukai type transform defined by the diagonal $\Delta_{\mathcal{W}} \hookrightarrow \mathcal{W} \times_{\mathcal{C}} \mathcal{W}$ (see [9] or Appendix B), so it is equivalent to the identity functor $id_{D^b(\mathcal{W})}$. Since i_0 is a closed embedding, $R^i i_{0\,*}(-) = 0$ for $i \neq 0$. Therefore $\Phi_0 \circ \Psi_0(\mathcal{F}) \cong \mathcal{F}$ for all objects $\mathcal{F} \in D^b(W)$.

To show $\Psi_0 \circ \Phi_0 \cong id$, we first note that Φ is an equivalence when Ψ is an equivalence. By a similar argument, one can show that $\Psi_0 \circ \Phi_0 \cong id$. q.e.d.

A. Perverse coherent sheaves

We give the definitions and related results of perverse coherent sheaves in this section. The main reference for this appendix is [8].

Let $f : X \to Y$ be a projective birational morphism between quasi-projective varieties. The following two assumptions are the same as in [8]:

(B.1) $\mathbf{R}f_*\mathcal{O}_X = \mathcal{O}_Y$.

(B.2) Every fiber of f is of dimension at most 1.

Any flopping contraction of a canonical threefold satisfies these two conditions.

We write $\mathcal{A} = D(X)$ and $\mathcal{B} = D(Y)$. By Proposition 2.3 in [8], we can identify $\mathcal{B}$ with a right admissible triangulated subcategory of $\mathcal{A}$. We thus have a semiorthogonal decomposition $(\mathcal{C}, \mathcal{B})$ where

$$\mathcal{C} = \mathcal{B}^{\perp} = \{E \in D(X) : \mathbf{R}f_*(E) = 0\}.$$

Lemma A.1. *An object $E \in D(X)$ lies in $\mathcal{C}$ precisely when its cohomology sheaves $H^i(E)$ lie in $\mathcal{C}$.*

Proof. This is Lemma 3.1 in [8]. The proof is an easy spectral sequence argument. The condition (B.2) is needed in the proof. q.e.d.

Now we can get a t-structure on $\mathcal{A}$ by gluing the t-structures on $\mathcal{B}$ and $\mathcal{C}$ (see [4] 1.4.8-10). The standard t-structure on $\mathcal{A}$ induces a t-structure $\mathcal{C}^{\leq 0} = \mathcal{C} \cap \mathcal{A}^{\leq 0}$ on $\mathcal{C}$. Shifting this by p and gluing it to the standard t-structure on $\mathcal{B}$ gives a new t-structure on $\mathcal{A}$.

This t-structure has the following properties:

$$\mathcal{A}_p^{\leq 0} = \{E \in \mathcal{A} : \mathbf{R}f_*(E) \in \mathcal{B}^{\leq 0} \text{ and } \mathrm{Hom}_{\mathcal{A}}(E, C) = 0 \text{ for all } C \in \mathcal{C}^{\geq p}\},$$
$$\mathcal{A}_p^{\geq 0} = \{E \in \mathcal{A} : \mathbf{R}f_*(E) \in \mathcal{B}^{\geq 0} \text{ and } \mathrm{Hom}_{\mathcal{A}}(E, C) = 0 \text{ for all } C \in \mathcal{C}^{\leq p}\}.$$

The heart of this t-structure is an abelian category $\mathrm{Per}^p(X/Y) = \mathcal{A}_p^{\leq 0} \cap \mathcal{A}_p^{\geq 0}$.
We shall only consider $p = -1$ and call this category $\mathrm{Per}\,(X/Y)$. Following Bridgeland, the objects of $\mathrm{Per}\,(X/Y)$ are called perverse coherent sheaves.

The next lemma gives an explicit description of $\mathrm{Per}(X/Y)$.

Lemma A.2. *An object E of $D(X)$ is a perverse sheaf if and only if the following three conditions are satisfied:*

(PS.1) $H_i(E) = 0$ *unless* $i = 0$ *or* 1.

(PS.2) $\mathbf{R}^1 f_* H_0(E) = 0$ *and* $\mathbf{R}^0 f_* H_1(E) = 0$.

(PS.3) $\mathrm{Hom}_X(H_0(E), C) = 0$ *for any sheaf C on X satisfying* $\mathbf{R}f_*(C) = 0$.

Proof. This is Lemma 3.2 in [8]. q.e.d.

Definition A.3. Two objects A_1 and A_2 of $D^b(X)$ are *numerically equivalent* if for any locally-free sheaf L on X we have $\chi(L, A_1) = \chi(L, A_2)$.

Definition A.4. An object F of $D(X)$ is a perverse ideal sheaf if there is an injection $F \hookrightarrow \mathcal{O}_X$ in the abelian category $\mathrm{Per}\,(X/Y)$. An object E of $D(X)$ is a perverse structure sheaf if there is a surjection $\mathcal{O}_X \to E$ in $\mathrm{Per}(X/Y)$. A perverse point sheaf is a perverse structure sheaf which is numerically equivalent to the structure sheaf of a point $x \in X$.

A perverse ideal sheaf F determines and is determined by a perverse structure sheaf E, which fit in an exact sequence in $\mathrm{Per}\,(X/Y)$

$$0 \longrightarrow F \longrightarrow \mathcal{O}_X \longrightarrow E \longrightarrow 0.$$

It turns out that a perverse ideal sheaf is a sheaf. We quote proposition 5.1 in [8].

Proposition A.5. *A perverse ideal sheaf on X is, in particular, a sheaf on X. A sheaf on X is a perverse ideal sheaf if and only if the following two conditions are satisfied:*

(PIS.1) *The sheaf $f_*(F)$ on Y is an ideal sheaf.*

(PIS.2) *The natural map of sheaves $f^*f_*(F) \to F$ is surjective.*

Let S be a scheme. Given a point $s \in S$, let $j_s : s \times X \to S \times Y$ be the embedding. As indicated in Bridgeland [8], a family of sheaves on X over S can be characterized as an object $\mathcal{F}$ of $D(S \times X)$ such that for each point $s \in S$ the object $\mathcal{F}_s = \mathbf{L}j_s^*(\mathcal{F})$ of $D(X)$ is a sheaf.

Following [8], we define the moduli functor of perverse sheaves.

Definition A.6. A family of perverse sheaves on X over a scheme S is an object $\mathcal{E}$ of $D(S \times X)$ such that for each point $s \in S$ the object $\mathcal{E}_s = \mathbf{L}j_s^*(\mathcal{F})$ of $D(X)$ is a perverse sheaf. Two such families $\mathcal{E}_1$ and $\mathcal{E}_2$ are equivalent if $\mathcal{E}_2 = \mathcal{E}_1 \otimes L$ for some line bundle pulled back from S. The moduli functor of perverse coherent sheaves assigns to each scheme S the set of equivalence classes of perverse coherent sheaves on $X \times S$.

The following theorem can be found in [8]:

Theorem A.7. *The functor which assigns to a scheme S the set of equivalence classes of families of perverse point sheaves on X over S is representable by a projective scheme $M(X/Y)$.*

The scheme $M(X/Y)$ has a distinguished irreducible component which is birational to Y. We shall call it $W(X/Y)$. When no confusion is possible, we denote it by W.

Remark A.8. In [8] Bridgeland proved the existence of a fine moduli space of perverse ideal sheaves when X and Y are projective varieties. We can generalize his existence result to quasi-projective varieties. A simple observation below shows how to weaken the projectivity assumption on Y.

Let $f : X \to Y$ be a projective morphism between two quasi-projective varieties satisfying the conditions (B.1) and (B.2). We can find a completion of $f : X \to Y$ as in the following diagram

$$\begin{CD} X @>{i_1}>> \overline{X} \\ @V{f}VV @VV{\overline{f}}V \\ Y @>{i_2}>> \overline{Y}. \end{CD}$$

The problem is that such a compactification may no longer satisfy conditions (B.1) and (B.2). But we can define another moduli functor which parameterizes the pairs of a sheaf F and a homomorphism $\alpha : F \to \mathcal{O}_{\overline{X}}$ satisfying conditions (PIS.1)-(PIS.2) in Proposition A.5 (see also Proposition 5.1 in [8]), and the condition that $\overline{f}_*(F)$ is the ideal of some point $y \in \overline{Y}$.

The proof of the existence of such a moduli space is the same as the proof of the existence of the moduli space of perverse point sheaves in [8]. When restricted to Y, this scheme is the moduli space of perverse point sheaves for $f : X \to Y$. It is also evident that this construction is local in Y.

B. Proof of Theorem 1.3

We sketch the proof of Theorem 1.3 in this appendix. The argument is taken from [9]. We adapt the notation from Theorem 1.3. The statement that W is a flop is an easy corollary of the result on the equivalence of derived categories (see [8]). We shall omit its proof.

Consider the diagram

$$\begin{array}{ccc} & W \times X & \\ {\scriptstyle \pi_W} \swarrow & & \searrow {\scriptstyle \pi_X} \\ W & & X. \end{array}$$

Let $\mathcal{P} \in D^b(W \times X)$ be the universal perverse point sheaf. We define a functor (using results in [22])

$$\Psi = R\pi_{X\,*}(\mathcal{P} \overset{L}{\otimes} \pi_W^*(-)) : D_{qc}(W) \to D_{qc}(X).$$

It turns out that Ψ sends $D^b(W)$ to $D^b(X)$ (see Step 1 below).

Step 1.

Each $\mathcal{P}_w$ has bounded homology sheaves. The variety X is nonsingular. These imply that $\mathcal{P}$ is of finite homological dimension. So we have $\Psi : D^b_c(W) \to D^b_c(X)$.

Step 2.

We define another functor $\Upsilon : D_{qc}(X) \to D_{qc}(W)$ by

$$\Upsilon(-) = [\mathbf{R}\pi_{W_*}(\mathcal{P}^\vee \otimes \pi_X^*(\omega_X)[n]) \overset{L}{\otimes} \pi_X^*(-)],$$

where $\mathcal{P}^\vee$ is the derived dual of $\mathcal{P}$. Note that this functor sends objects in $D_c^b(X)$ to $D_c^b(W)$ since $\mathcal{P}^\vee$ is of finite homological dimension. We now restrict this functor to D_c^b. Then $\Upsilon : D_c^b(X) \to D_c^b(W)$ is left adjoint to $\Psi : D_c^b(W) \to D_c^b(X)$ as shown in [9] (p. 16). The composite functor $\Upsilon \circ \Psi$ is given by $\mathbf{R}\pi_{2_*}(\mathcal{Q} \overset{L}{\otimes} \pi_1^*(-))$, where $\pi_1, \pi_2 : W \times W \to W$ are the projections and $\mathcal{Q}$ is some object of $D_c^b(W \times W)$.

If $i_w : w \times W \hookrightarrow W \times W$ be the embedding, then $\mathbf{L}i_w^*(\mathcal{Q}) = \Upsilon\Psi\mathcal{O}_w$. We have the following isomorphisms

$$\begin{aligned}
\mathrm{Hom}^i_{D(W\times W)}(\mathcal{Q}, \mathcal{O}_{w_1,w_2}) &= \mathrm{Hom}^i_{D(W)}(\Upsilon\Psi\mathcal{O}_{w_1}, \mathcal{O}_{w_2}) \\
&= \mathrm{Ext}^i_X(\Psi\mathcal{O}_{w_1}, \Psi\mathcal{O}_{w_2}) \\
&= \mathrm{Ext}^i_X(\mathcal{P}_{w_1}, \mathcal{P}_{w_2}).
\end{aligned}$$

Each $\mathcal{P}_w$ is simple, so its support is connected and since $\mathbf{R}f_*(\mathcal{P}_w) = \mathcal{O}_y$, where $y = g(w)$, it follows that $\mathcal{P}_w$ is supported on a fiber of f over y. Since f is crepant, we have $\mathcal{P}_w \otimes \omega = \mathcal{P}_w$. For distinct w_1, w_2, Serre duality together with Lemma 3.6 in [8] implies that $\mathrm{Hom}^i_{D_c}(\mathcal{P}_{w_1}, \mathcal{P}_{w_2}) = 0$ unless $g(w_1) = g(w_2)$ and $1 \le i \le n-1$. Since $X \to Y$ is crepant, it follows that $\mathcal{P}_w \otimes \omega_X = \mathcal{P}_w$.

Step 3.

We proved in Step 2 that $\mathrm{h.d.}(\mathcal{Q}) \le (n-1) - 1 = n-2$ when restricted to $W \times W - \Delta_W$. We know that dim $(W \times_Y W) \le n+1$ by assumption, and $\mathrm{Supp}(\mathcal{Q})$ is contained in $W \times_Y W$. Since we have $\mathrm{codim}(\mathcal{Q}) \ge n-1$, the intersection theorem implies $\mathcal{Q} \cong 0$ outside the diagonal.

Fix a point $w \in W$, put $E = \Upsilon \circ \Psi(\mathcal{O}_w)$. We prove above that E is supported at the point w.

Claim B.1. $H_0(E) = \mathcal{O}_w$.

The proof of this claim can be found in [9] (p. 18). Corollary 5.3 in [9] then implies that $E \cong \mathcal{O}_w$ and W is nonsingular. Applying Theorem 2.3 in [9], it follows that $\Psi : D_c^b(W) \to D_c^b(X)$ is an equivalence of derived categories. The essence of Theorem 2.3 in [9] is the using of Serre duality and adjoint pairs.

We remark that their argument also shows that

$$\Psi : \mathrm{Ext}^i_W(\mathcal{O}_{w_1}, \mathcal{O}_{w_2}) \to \mathrm{Ext}^i_X(\mathcal{P}_{w_1}, \mathcal{P}_{w_2})$$

are isomorphisms for all i (see [9] p. 18), from which one can prove that W is actually a connected component of $M(X/Y)$.

Step 4.

The functor $\Psi : D^b_c(W) \to D^b_c(X)$ has a right adjoint $\Phi : D^b_c(X) \to D^b_c(W)$, which is also a Fourier-Mukai type transform. The reader can see [9] for an explicit formula. We show that Ψ is fully faithful in this step. It suffices to show that $\Phi \circ \Psi \cong id$.

The composition functor $\Phi \circ \Psi$ is $\mathbf{R}\pi_{2_*}(\mathcal{Q}_1 \overset{L}{\otimes} \pi_1^*(-))$ where $\pi_1,\ \pi_2$ are the projections $W \times W \to W$ and $\mathcal{Q}_1$ is some object of $D(W \times W)$. It suffices to show that $\mathcal{Q}_1$ is quasi-isomorphic to $\mathcal{O}_{\Delta_W}$. We have $\mathbf{L}i_w^*(\mathcal{Q}_1) = \Phi \circ \Psi(\mathcal{O}_w)$. By an argument similar to the one given in Step 3, we have $\Phi \circ \Psi(\mathcal{O}_w) = \mathcal{O}_w$ for all w. This shows that $\mathcal{Q}_1$ is actually the push-forward of a line bundle on W to the diagonal $W \times W$. So $\Phi \circ \Psi$ is just twisting by L. To prove $\mathcal{Q}_1$ is quasi-isomorphic to $\mathcal{O}_{\Delta_W}$, it remains to show L is trivial.

There is a natural transform $\varepsilon : id \to \Phi \circ \Psi$, which gives a commutative diagram for every w:

$$\begin{array}{ccc}
\mathcal{O}_W & \xrightarrow{\varepsilon(\mathcal{O}_W)} & L \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle L \otimes a} \\
\mathcal{O}_w & \xrightarrow{\varepsilon(\mathcal{O}_w)} & \mathcal{O}_w
\end{array}$$

where a is nonzero. Since ε is an isomorphism on the subcategory $D_c(W)$, it implies $\epsilon(\mathcal{O}_W)$ is an isomorphism. This shows that $\mathcal{Q}_1$ is quasi-isomorphic to $\mathcal{O}_{\Delta_W}$.

Step 5.

By Lemma 2.1 in [9], the statement that the Fourier-Mukai type transform Ψ is an equivalence of derived categories follows from the following statement

$$\Phi(E) \cong 0 \Longrightarrow E \cong 0 \ \ \forall E \in D(X).$$

A proof of this statement can be found in Step 9 in [9].

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